Stochastic models of Markets and Option Pricing using the Black and Scholes Model

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In mathematics a financial market is composed by:

- a stochastic representation of the events and information (market place)
- the set of traded assets (i.e. financial products whose values are quoted)

**Definition.** A market place of duration $T > 0$ is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathcal{F}_t$ such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$.

- $\mathcal{F}_t$ is information in finance

**Definition.** An asset price or return is any positive valued stochastic process which belongs to $\mathcal{L}^p(\emptyset, T)$ with $p \in [0, \infty]$

**Definition.** A (finite-dimensional) market $\mathcal{M}(X_t)$ is a couple composed by a market place and a set $X_t = (X_t^0, X_t^1, ..., X_t^N)$ of assets' returns which are named primary. The linear span of the primary assets' returns is said market space. Conventionally $X_t^0 \in \mathcal{L}^\infty(\emptyset, T)$ stands for a money market account and the returns $X_t^j = X_t^j/X_t^0 = e^{-rt}X_t^j$ $(j = 1, ..., N)$ are named actualised prices.
We will see that **actualised prices**, under certain conditions, are **martingales**.

**Definition.** A process $X_t$ is adapted to a filtration $\{F_t : t \in [0,T]\}$ if it is $F_t$-measurable $\forall t$.

**Definition.** Let $(\Omega, F, P)$ be furnished by the filtration $\{F_t : t \in [0,T]\}$. Moreover, let $X_t$ be a stochastic process. A Martingale is any $F_t$-adapted process such that:

\[
\mathbb{E}[|X_t|] < \infty \quad \forall t,
\]

\[
\mathbb{E}[X_s | F_t] = X_t \quad \forall s \geq t.
\]
Types of assets in the market: riskless and risky

- **Definition.** A *money market account* or a *bond* is a "riskless asset", whose price $B \in \mathcal{L}^\infty$ is driven by the equation

$$B_t = \exp \left( \int_0^t r_\tau \, d\tau \right).$$

where $r \in \mathcal{L}^\infty_+$ is the instantaneous **interest rate**.

$B$ changes according to the ordinary differential equation:

$$\frac{dB}{dt} = rB.$$

- **Definition.** A *stock* is a "risky asset" whose price $S \in \mathcal{L}^2(0,T)$ is assigned by means of a **stochastic forward** differential equation.

The specific choice of the process driving stock returns is what distinguishes one market model from another.
In the Black and Scholes model a diffusion model is proposed:

\[ dS_t = S_t[\alpha dt + \sigma dW_t] \]

- \( \alpha \) is the **drift**
- \( \sigma \) the **volatility**
- \( W_t \) is a **standard Brownian motion**

Brownian motion is used in applications to finance because a quantity which behaves as a standard Brownian motion can only assume **strictly positive values**.

**Blue line:**
\( \alpha = 1, \ \sigma = 0.2 \)

**Green line:**
\( \alpha = 0.5, \ \sigma = 0.5 \)
The Black and Scholes model had great fortune in the Financial World due to:

✓ the easiness to use it
✓ the possibility to deduce the price of the options in closed form in some cases (EU Options).

But the resultant forecasts encounter relevant disagreements with the real world.

⚠️ Volatility constant w.r.t. time! ⚠️

Deducing $\sigma$ backwards from the asset price the resulting implied volatility depends on the option’s maturity

$\sigma$ calculated at different instants of times may assume different values
Merton’s Jump Diffusion Model

To reduce discrepancies

Merton's Model, based on a jump diffusion process:

\[ dS_t = S_t[\alpha dt + \sigma dW_t + \gamma dN_t]. \]

- \( \gamma \) is the size of jump
- \( N \) is a Poisson process

\[ \begin{align*}
\alpha &= 0.18 \\
\sigma &= 0.73 \\
\gamma &= 3 \\
T &= 1 \text{ (year)}
\end{align*} \]
A **Trader** is a financial operator who buys and sells financial instruments.

The **Portfolio** of a trader is composed by the titles (both risky and riskless) and the amount of them that he owns.

**Definition.** A **financial strategy** or **dynamic portfolio** in the market $\mathcal{M}(X.)$ is a $\mathcal{F}_t$-predictable process $\Delta.$ with value in $\mathbb{R}^{N+1}$ such that $\Delta^i_t \in \mathcal{L}^p_t(0,T).$ Here $\Delta^i_t$ stands for the number of shares of the $i^{th}$ asset held at time $t$.

A strategy is **self-financing** if its total wealth $\Delta_t \cdot X_t$ may be read as a differential curve on the market space, namely if $d(\Delta_t \cdot X_t) = \Delta_t \cdot dX_t$ in the following sense:

$$\Delta_t \cdot X_t = \Delta_0 \cdot X_0 + \int_0^t \Delta_r \cdot dX_r.$$  

Here the integral has to be intended in **Ito's sense**.

A strategy is **admissible** if it is self-financing and has nonnegative total wealth.
For $0 \leq t_0 \leq t_1 \leq \ldots \leq t_T$, a sequence of random variables $(X_t)_t$ is $\mathcal{F}_t$ predictable if:

- $X_{t_0}$ is $\mathcal{F}_{t_0}$-measurable
- $\forall t \geq 0$ and $\forall s \geq 0$ with $t \geq s$: $X_t$ is $\mathcal{F}_s$-measurable.

The assumption of predictability is natural: $\Delta_t$ is $\mathcal{F}_s$-measurable (with $t \geq s$), the investor has to establish the quantities before he sees the values of the prices at time $t$.

No condition imposed about the sign of $\Delta^i_t$: $\Delta^i_t < 0$, for instance, means that the agent is short selling $|\Delta^i_t|$ shares of the $i^{th}$-asset at time $t$.

Self financing strategy requires no extra costs during trading except for the initial capital.

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Itô's lemma is an identity which finds the differential of a time-dependent function of a process. It is the equivalent form of the Taylor series expansion of the function up to its second derivatives and retaining terms up to first order in the time increment and second order in stochastic processes.

Let $X_t, t \in [0, T]$ a Itô process, i.e. an adapted stochastic process such that:

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s$$

where $K_s$ and $H_s$ are two uniquely identify functions sucht that:

$$\int_0^T |K_s| ds < \infty, \quad \int_0^T H_s^2 ds < \infty \text{ P.a.s.}$$

Let hence $f : [0 + \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ be a function in $C^2$, then the stochastic process $Y_t = f(X_t, t)$ is a Itô’s process determined by:

$$f(X_t, t) = f(X_0, 0) + \int_0^t f_s(X_s, s) ds + \int_0^t f_x(X_s, s) K_s ds + \int_0^t f(X_s, s) H_s dW_s + \frac{1}{2} f_{xx}(X_s, s) H_s^2 ds$$
Definition. A contingent claim or derivative on the underlying assets \(X^1, ..., X^N\) is a couple \((T, G)\) where \(T \geq 0\) is called time of expiration or maturity and \(G \in C([0, \infty)^N) \cap W_{loc}^{1, \infty}(0, \infty)^N\) and \(G \geq 0\) is the liquidation value or payoff.

\(W_{loc}^{1, \infty}\) is the set of the locally Lipschitz functions continuous in \((0, \infty)^N\).

The payoff is the loss of money that the writer suffers if the option is exercised.

Derivatives may be used essentially for two different reasons:

- to cover the risk (doesn't mean to "hide" it, but to reduce it.);
- speculative purpose;

there are many kinds of derivative instruments; in this framework I will treat only one type of derivative instruments:

Options
Option pricing theory is the theory of how options are valued in the market.

- **Options** are contracts sold by one party (the writer) to another one (the holder) that give the holder the right, but not the obligation, to buy or sell a security or other financial asset **at an agreed-upon price** $K$ (strike price) **during a certain period** of time or on a **specific date** $T$ (maturity/exercise date).

- An option has a cost ➤ the price of an option is called **premium**

- The premium is paid when the contract is initiated.

- Options can be divided in **two** families:
  - **European**: can only be exercised at maturity $T$.
  - **American**: can be exercised at any moment before maturity.
**Call and Put Options:** The decision of the holder to either exercise or not the contract depends on the comparison between $S_T$ and $K$.

**Payoff value for call option:** $G = (S_T - K)_+ = \max(S_T - K, 0)$

- $S_T > K$ : Option exercised;
- $S_T < K$ : Option NOT exercised.

**Payoff value for put option:** $G = (K - S_T)_+ = \max(K - S_T, 0)$

- $S_T > K$ : Option NOT exercised;
- $S_T < K$ : Option exercised.
Two essential questions:

1. Does a deterministic function $U(X,t)$ that gives the value of the derivative, exist (at a time $t < T$)? 
   (Problem of pricing);

2. Does a deterministic hedging strategy exist in order to allow the writer to minimize the risk associated to his options? 
   (Problem of the hedging).

Assumption 1: neglection of frictions, i.e.
- we do not consider transaction costs
- borrowing from bank and short selling are indefinitely permitted
- assets are divisible

Assumption 2:
- Markets with no arbitrage opportunities
- Complete Markets
An **arbitrage** is the simultaneous purchase and sale of an asset to profit from a difference in the price.

The **absence of arbitrage** or "no free-lunch" is a financial principle which states that there is no opportunity to make an instantaneous risk-free profit.

**1 Theorem of Asset Pricing.** *A market has no arbitrage opportunities* $\iff \exists$ *a probability measure* $\mathbb{P}^* \sim \mathbb{P}$ *under which the actualized assets' price are martingales.* Such a probability is usually called **risk-neutral** or equivalent **martingale measure**.

**Remark.** *Two probability measures equivalent* $\mathbb{P}^* \sim \mathbb{P}$ *if they share a common* $\Omega$, *a common* $\mathcal{F}$ *sigma-algebra of events and, for any* $A$:

$$\mathbb{P}(A) > 0 \iff \mathbb{P}^*(A) > 0$$

(*i.e. they define the same null sets*)
Absence of Arbitrage & Hedging Strategy

The absence of arbitrage plays a crucial role because it allows to price contingent claim by means of the so called hedging strategy.

➢ **Definition.** An admissible strategy $\Delta$ is a hedging strategy for the European contingent claim $(T, G)$ if: $\Delta_T \cdot X_T = G(X_T)$

with probability 1.

A $\Delta$-hedging strategy hence **mitigates the financial risk of an option** by hedging against price changes in its underlying.
Pricing by Arbitrage

We suppose that market $\mathcal{M}(X)$ is assigned by means of a forward stochastic differential system:

$$X_t^i = X_0^i + \int_0^t \alpha^i_\tau d\tau - \sum_{j=1}^P \int_0^t \Delta \cdot \beta_j^i \, dE^j_\tau.$$ 

where $\alpha^i$ and $\beta_j^i$ are deterministic coefficients and $E$ is a stochastic process, standing for a basis of randomness.

The question of pricing by arbitrage may be formulated as the problem of finding the minimal solution of a backward-forward stochastic differential equation.

Corollary 1. Let $(T,G)$ be a European contingent claim in the market $\mathcal{M}(X)$. We suppose that there exists an hedging strategy $\Delta$ such that it's total wealth $Y = \Delta \cdot X$. is the minimal solution of the backward stochastic differential equation

$$Y_t = G(X_T) - \int_t^T \Delta \alpha \, d\tau - \sum_{j=1}^P \int_t^T \Delta \beta_j \, dE^j_\tau$$
Corollary 2. Let $\mathcal{M}(X)$ be the market without arbitrage opportunities and $(T,G)$ a contingent claim. We assume that there exist a hedging strategy $\Delta_\cdot$ for $(T,G)$ and a deterministic continuous function $U(X,t)$ such that $U(X_t,t) = Y_t = \Delta_t \cdot X_t \forall t$. If $U$ has the following regularity properties:

$$U \in \mathcal{L}^\infty(0,T; W^{2,\infty}(0,\infty)^N)$$

$$\delta_t U \in \mathcal{L}^\infty(0,T; \mathcal{L}^\infty(0,\infty)^N),$$

Then $Y_\cdot$ is exactly the arbitrage value of $(T,G)$.
With the absence of arbitrage plus Ito’s calculus:

Option pricing and Hedging \( \rightarrow \) Cauchy problem for a deterministic function differential operator.

\( u(S, t) \) is the value of a contingent claim at \( t \) of \( S \).

The hedging strategy is given deterministically and explicitly in terms of \( u \).
Complete Markets

- **Definition.** A market is complete if all contingent claims have a hedging strategy.

2 Theorem of the Asset Pricing. *Assuming that the market is without arbitrage opportunities, it is also complete if and only if the equivalent martingale measure is unique.*

Assumption of market completeness makes it easier to elaborate models for computing options prices and portfolios.

Pricing by arbitrage leads to a deterministic differential problem by mean of virtual hedging combined with Ito's calculus.

If the market is **complete**, this technique bring to a **unique and well defined** value for any contingent claim.
The Black and Scholes model concerns the primary market generated by one risky stock, in continuous time and does not pay dividends.

**Assumptions:**

1. The stock's price follows the random walk:
   \[ dS_t = S_t [\alpha dt + \sigma dW_t], \]
2. the interest rate \( r \) is a known constant,
3. the are no transaction costs associated with hedging a portfolio,
4. short selling is permitted,
5. assets are divisible,
6. BS markets are **arbitrage-free**.

If the Market is also complete, we substitute \( \alpha \) with \( r \), obtaining:

\[ dS_t = S_t [r dt + \sigma dW_t] \]
Proposition. The Black and Scholes' market is complete. Moreover the arbitrage price of the European contingent claim \((T, G)\) is given by the solution of the final value problem on \((0, \infty) \times (0, T)\):

\[
\begin{aligned}
-\delta_t U &= \frac{1}{2} \sigma^2 S^2 \delta_{ss} U + r S \delta_s U - r U, \\
U(S, T) &= G(S).
\end{aligned}
\]

and the hedging strategy is given deterministically as a function of \((S_T, t)\) by:

\[
\Delta^0_t = e^{-rt}[U(S_t, t) - S_t \delta_s U(S_t, t)],
\]

\[
\Delta_t = \delta_s U(S_t, t).
\]

Black & Sholes operator: \[-\delta_t U = \frac{1}{2} \sigma^2 S^2 \delta_{ss} U + r S \delta_s U - r U.\]

The hedging strategy \((\Delta^0, \Delta)\) is admissibile for the BS model.
Using the **probabilistic interpretation** of the PDE from **Feynman-Kac’s formula**, the solution is of the form:

$$U(S_t, t) = e^{-r(T-t)} \mathbb{E}[G(S_T)|\mathcal{F}_t]$$

By **1&2TAP** $\mathbb{P}^* \sim \mathbb{P}$ and under $\mathbb{P}^*$ we have $\tilde{S}_t = e^{-rt}S_t$ and $\alpha = r$ leading to:

$$dS_t = S_t[r dt + \sigma dW_t]$$

Rewriting the solution under the risk free probability measure:

$$U(S_t, t) = e^{-r(T-t)} \mathbb{E}^*[G(S_T)|\mathcal{F}_t]$$

For a **Call Option**, substituting $G(S_T) = (S_T - K)_+$ and computing the Conditional Expectation, we obtain the **price**:

$$U(S_t, t) = S_t \Phi(d_1) - Ke^{-r(T-t)} \Phi(d_2)$$

where:

$$d_1 = \frac{\log(S_t/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} \quad \Phi \text{ d.f. } Z \sim \mathcal{N}(0, 1)$$

$$d_2 = d_1 - \sigma \sqrt{T-t}$$

In the same way, for a **Put Option** ($G(S_T) = (K - S_T)_+$), the **price** is:

$$U(S_t, t) = Ke^{-r(T-t)} \Phi(-d_2) - S_t \Phi(-d_1)$$
% Stock price simulation
function S_t=randomwalk(S_0, r, sigma, T, N)
    t=linspace(0,T,N+1);
dt=T/N;
    S_t=zeros(N+1,1);
    S_t(1)=S_0;
for i=2:N+1;
    W=randn();
    S_t(i)=S_t(i-1)+r*S_t(i-1)*dt+sigma*S_t(i-1)*sqrt(dt)*W;
end
plot(t,S_t);
title('Asset Price Plot');
end

% European Call and Put Option Price & Delta Hedging using
% Black-Scholes formula
function S_t=randomwalk(S_0, r, sigma, T, N);
    bsc1 =zeros(N+1,1);
    bsc2 =zeros(N+1,1);
    cdelta=zeros(N+1,1);
pdelta=zeros(N+1,1);
for i=1:N+1;
    tau(i)=sqrt(T-t(i));
    d1(i)=(log(S_t(i)./K)+(r+0.5.*sigma^2)*T-t(i))/(sigma.*tau(i));
    d2(i)=d1(i)-sigma.*tau(i);
    bsc1(i)=S_t(i).*normcdf(d1(i))-K.*exp(-r.*T-t(i))*normcdf(d2(i));
    bsc2(i)=K.*exp(-r.*T-t(i))*normcdf(-d2(i))-S_t(i).*normcdf(-d1(i));
    cdelta(i)=0.5.*erfc(-d1(i)/sqrt(t(i)));
    pdelta(i)=normcdf(d1(i));
pdelta(i)=-normcdf(-d1(i));
end
    echo on;
plot(t, S_t, 'k'), title('Stock Price');
subplot(2,2,1), plot(t, bsc1, 'b'), title('Call Price');
subplot(2,2,2), plot(t, bsc2, 'r'), title('Put Price');
subplot(2,2,3), plot(t, cdelta, 'g'), title('Call Delta');
subplot(2,2,4), plot(t, pdelta, 'y'), title('Put Delta');
SIMULATION OF A STOCK PRICE, RELATED CALL-PUT PRICES & DELTA HEDGING PLOTS ON MATLAB

- $S_0 = 50$
- $K = 50$
- $\sigma = 0.5$
- $r = 0.05$
- $T = 1$ (year)
- $N = 100$

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Through the Black and Sholes model it is possible to obtain the value of the price of a European contingent claim in a close form.

From $U(X,t)$ it is possibile to determine an hedging strategy that allows the writer to minimize the risk associated to its options.
Future Objectives

➢ Study the pricing of *American Options*.

➢ Options pricing and hedging using other models (Merton, Montecarlo, CRR...).
References

➢ C. Palandra: *Probability and Finance - Bachelor's Degree Thesis*, University of Rome "Tor Vergata", 2006.
Thank you for your attention!